

A non-commutative generalisation of Thurston's gluing equations

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Abstract

In his famous Princeton Notes, Thurston introduced the so-called gluing equations defining the deformation variety. Later, Kashaev defined a non-commutative ring from H-triangulations of 3-manifolds and observed that for trefoil and figure-eight knot complements the abelianization of this ring is isomorphic to the ring of regular functions on the deformation variety, [2, 3, 5]. In this paper, we prove that this is true for any knot complement in a homology sphere. We also analyse some examples on other manifolds.

1 Introduction

In his famous Princeton notes [13], Thurston introduced the following gluing equations. Starting from an ideal triangulation X of a cusped manifold, $M \setminus K$ assign to each edge of each tetrahedron a shape parameter $z \in \mathbb{C}$ so that:

- if z_1, z_2, z_3 are three shape parameters counter-clockwisely ordered around a vertex of a tetrahedron, then $z_1 z_2 z_3 = -1$ and $z_2 - z_1 z_2 = 1$;
- for each edge E of X , let z_1, \dots, z_n be the shape parameters of all the edges which are identified to E , then $\prod_{i=1}^n z_i = 1$.

This defines a set of polynomial equations, hence an affine variety in $(\mathbb{C} \setminus \{0, 1\})^{6n}$, where n is the number of tetrahedra in the triangulation. This affine variety is called *deformation variety*. We denote by R_I its associated ring of regular functions. The deformation variety may be empty for some triangulations. In that case, R_I is defined to be the ring with one element.

Since then, the gluing equations have been extensively studied, see for example [6, 7, 9, 10, 11, 12].

Later, Kashaev introduced Δ -groupoids [2] and B' -rings associated to ideal triangulations of manifolds and computed them for the trefoil and the figure eight knot complements [5].

Then, in [3], Kashaev introduced the ring \tilde{R}_H , the abelianisation of which is studied in this article. This ring is associated to a particular 1-vertex H-triangulation of a pair (M^3, K) (where M^3 is a connected, oriented, closed 3-manifold and K is a knot) to which naturally corresponds an ideal triangulation of the complement of K in M^3 . This will be precised in section 2. In the case

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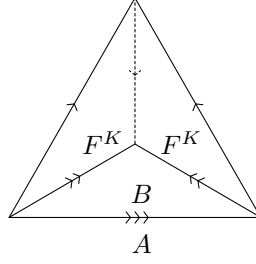


Figure 1: Example of a distinguished tetrahedron T^K . Its distinguished edge K is dotted.

of the trefoil knot, \tilde{R}_H is abelian and isomorphic to R_I . In the case of the figure-eight knot, \tilde{R}_H is not abelian, but its abelianization is isomorphic to R_I .

We prove in the following that for any knot embedded in a homology 3-sphere there exists an isomorphism between the abelianization of the ring \tilde{R}_H defined from particular H-triangulations and the ring of regular functions on the deformation variety of a corresponding ideal triangulation. This isomorphism is explicitly constructed in section 3 and the theorem proved in section 4. In section 5 we give an example of a knot embedded in the 3-sphere and examples of knots embedded in other manifolds.

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2 Preliminaries

2.1 Triangulations

Let M be a connected, oriented, closed 3-manifold. A triangulation Δ of M is defined to consist of a pairwise disjoint union of oriented euclidian tetrahedra $\Delta^3 = \bigsqcup_{i=1}^n \Delta_i^3$, together with a collection Φ of orientation-reversing affine isomorphisms pairing the faces of the tetrahedra in Δ^3 , so that M is homeomorphic to the identified space Δ^3/Φ . In the following, for $k \in \mathbb{N}$, Δ^k will denote the set of cells of dimension k of the triangulation Δ .

An H-triangulation is a pair (Δ, K) , where Δ is a triangulation of a 3-manifold M and K is a hamiltonian subcomplex of the 1-skeleton of Δ . In the following, we will consider only *particular* H-triangulations: we will assume that Δ has only one vertex, and the hamiltonian subcomplex is given by a single edge K , which is contained in a single face, F^K of Δ , obtained from a single tetrahedron T^K of Δ , glued to itself as a closed book by Φ along its two faces. The edge K represents a knot embedded in M . Such a H-triangulation of (M, K) will be denoted \mathcal{T}_H . In order to stress the "H-nature" of \mathcal{T}_H , the set of k dimensional cells of \mathcal{T}_H will be denoted $\Delta^{k, \mathbf{H}}$ and the set of face pairings, $\Phi^{\mathbf{H}}$. We will denote $p_H : \Delta^{3, \mathbf{H}} \rightarrow \Delta^{3, \mathbf{H}}/\Phi^{\mathbf{H}}$ the identification projection. K , F^K , T^K will respectively be called *distinguished edge*, *distinguished face* and *distinguished tetrahedron* (see figure 1).

Proposition 1 *For any knot K in \mathbb{S}^3 , there exists a particular H -triangulation of (\mathbb{S}^3, K) .*

Proof There exist algorithms for one vertex H -triangulations of couples (\mathbb{S}^3, K) . See, for example, [4], section 4. A slight modification of this algorithm allows one to get a particular H -triangulation: at the last stage of the decomposition of the 3-cell onto several tetrahedra, one has to extract a distinguished tetrahedron, and the previous steps of the algorithm always make this possible. \square

From \mathcal{T}_H , one gets a cell decomposition of another manifold by removing an open neighbourhood of the vertex. This is equivalent to truncating the tetrahedra, so that the new 3-cells are bounded by *triangular* and *hexagonal* faces. The triangular faces are bounded by *short edges*, and hexagonal faces are bounded by short and *long edges* which are remnant of the edges of \mathcal{T}_H . The obtained manifold is the complement in M of an open ball. Such a cell decomposition will be denoted $\overline{\mathcal{T}_H}$ and the set of k dimensional cells in $\overline{\mathcal{T}_H}$ will be denoted $\overline{\Delta^{k,H}}$. Figure 2 is an example of a truncated tetrahedron. We will denote $\mathcal{H} \subset \overline{\Delta^{2,H}}/\Phi^H$ and $\tilde{\mathcal{H}} \subset \overline{\Delta^{2,H}}$ the sets of non distinguished hexagonal faces, \mathcal{T} the set of triangular faces disjoint from the distinguished edge, $\mathcal{L} \subset \overline{\Delta^{1,H}}/\Phi^H$ and $\tilde{\mathcal{L}} \subset \overline{\Delta^{1,H}}$ the sets of non distinguished long edges, $\mathcal{S} \subset \overline{\Delta^{1,H}}/\Phi^H$, and $\tilde{\mathcal{S}} \subset \overline{\Delta^{1,H}}$ the sets of short edges disjoint from the knot.

An ideal triangulation of a connected, oriented, cusped manifold N , is defined to consist of a pairwise disjoint union of oriented euclidian tetrahedra $\Delta^{3,I} = \bigsqcup_{i=1}^n \Delta_i^{3,I}$, together with a set Φ^I of orientation-reversing affine isomorphisms pairing the faces of the tetrahedra in $\Delta^{3,I}$ so that $N = (\Delta^{3,I} \setminus \Delta^{0,I})/\Phi^I$. Such a triangulation will be denoted \mathcal{T}_I . We will denote $p_I : \Delta^{3,I} \setminus \Delta^{0,I} \rightarrow (\Delta^{3,I} \setminus \Delta^{0,I})/\Phi^I$ the identification projection. In the following, we will consider a knot K embedded in a 3-manifold M and ideal triangulations of $N = M \setminus K$, such that $\Delta^{3,I}/\Phi^I$ is a pseudomanifold having one singular point, its only vertex, corresponding to K . The tetrahedra of \mathcal{T}_I can be seen as hyperbolic ideal tetrahedra with vertices at infinity.

Let us remark that from a particular H -triangulation of a pair (M, K) , one can get an ideal triangulation of $M \setminus K$ by collapsing the distinguished edge to a point in such a way that the distinguished face F^K is collapsed to an edge while the two other faces of the distinguished tetrahedron (not bounded by distinguished edge) are identified with each other. For example, in figure 1, the edges with simple and double arrows are identified with each other; face F^K collapses and faces B and A are identified with each other. Then, each cell of \mathcal{T}_H different from the distinguished ones have a canonical corresponding cell in \mathcal{T}_I .

One can find more information on triangulations of 3-manifolds in [1, 8].

From an ideal triangulation of a knot complement in a 3-manifold, one can construct a ring R_I as in the introduction.

2.2 Definition of R_H

According to [3], the ring \tilde{R}_H is defined from $\overline{\mathcal{T}_H}$ with oriented short edges by the following presentation. The set of generators is given by associating to each oriented short edge $e \in \mathcal{S}$, disjoint with the distinguished edge, a pair of generators (u_e, v_e) ; and the set of relations:

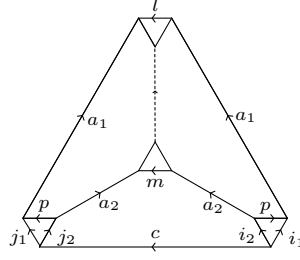


Figure 2: Truncated distinguished tetrahedron

- if \bar{e} is the edge e with opposite orientation, $u_{\bar{e}} = u_e^{-1}$ and $v_{\bar{e}} = -u_e^{-1}v_e$;
- *hexagonal face relation*: if \check{e} is the unique oriented short edge such that it belongs to the same hexagonal face as e , and the terminal points of e and \check{e} form the boundary of a long edge, then $u_{\check{e}} = v_e$ and $v_{\check{e}} = u_e$;
- *triangular face relations*: if e_1, e_2, e_3 are cyclically oriented short edges constituting the boundary of a triangular face, then $u_{e_1}u_{e_2}u_{e_3} = 1$ and $u_{e_1}u_{e_2}v_{e_3} + u_{e_1}v_{e_2} + v_{e_1} = 0$.

One can easily check that there exists a representation of the knot group onto $GL(2, \tilde{R}_H)$ by associating the matrix $\begin{pmatrix} u_e & v_e \\ 0 & 1 \end{pmatrix}$ to each oriented short edge disjoint from the distinguished edge and the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ to each (oriented) long edge different from the distinguished one.

Note that the ring \tilde{R}_H is not necessarily commutative. We will denote by R_H the abelianization of \tilde{R}_H .

Lemma 2 *Let the truncated distinguished tetrahedron be labelled as in figure 2, then $u_l = u_m$, $u_p = v_m^{-1}v_l$, $v_p = 0$.*

Proof Let us write the two triangular face relations where all generators with indices i_1, i_2, j_1, j_2 are expressed in terms of (u_l, v_l) and (u_m, v_m) through the use of the hexagonal faces relations.

For the left hand triangle:

$$1 = v_m u_p v_l^{-1} \quad (1)$$

$$0 = -v_m u_p v_l^{-1} u_l + v_m v_p + u_m \quad (2)$$

and the right hand triangle:

$$1 = u_m^{-1} v_m u_p v_l^{-1} u_l \quad (3)$$

$$0 = -u_m^{-1} v_m u_p (v_l^{-1} u_l) u_l^{-1} + u_m^{-1} v_m v_p + u_m^{-1} \quad (4)$$

Equations (1) and (3) imply that $u_m = u_l$. Then (2) implies that $v_p = 0$ because v_m is invertible. \square

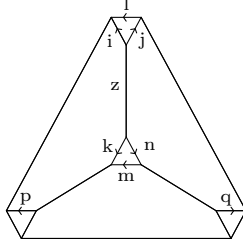


Figure 3: $f(z) = v_i v_j^{-1}$

3 A ring homomorphism from R_I to R_H

We define a ring homomorphism $f : R_I \rightarrow R_H$ as follows. Let E be an edge of a tetrahedron of the ideal triangulation with shape parameter z . Let us still call E the corresponding long edge in $\tilde{\mathcal{L}}$. We choose a boundary point of E in $\overline{\Delta^{0,\mathbf{H}}}$, and let e_i and e_j be the short edges in \tilde{S} sharing this point at their origins and such that E, e_i, e_j are clockwise ordered. Let (u_i, v_i) and (u_j, v_j) be the couples of generators in R_H assigned to e_i and e_j . We define, $f(z) = v_i v_j^{-1}$ (see figure 3).

Remark 3 Keeping the notation of figure 3, f can be equivalently defined as $f(z) = u_p u_q^{-1}$.

Let us check that f is a well defined ring homomorphism.

First, the definition of f does not depend on the choice of the boundary point of E : if $e_k = \overline{e_i}$ and $e_n = \overline{e_j}$, then $v_i = v_k^{-1}$ and $v_j = v_n^{-1}$ so that $v_i v_j^{-1} = v_n v_k^{-1}$.

Let E_1, E_2, E_3 be three long edges counter-clockwisely ordered around a vertex of a tetrahedron in $\Delta^{3,\mathbf{I}}$, with shape parameters z_1, z_2, z_3 , and let e_i, e_j, e_k be three short edges bounding a triangular face obtained after truncation along the corresponding vertex and such that E_1, e_i, e_j and $E_2, \overline{e_j}, e_k$ are clockwise ordered. Then, the triangular face relations are $u_j u_k u_i^{-1} = 1$ and $-v_i + u_j v_k + v_j = 0$.

So,

$$f(z_1)f(z_2)f(z_3) = (v_i v_j^{-1})(-u_j^{-1} v_j v_k^{-1})(u_k^{-1} v_k v_i^{-1} u_i) = -u_j^{-1} u_k^{-1} u_i = -1$$

and

$$f(z_2) - f(z_1)f(z_2) = -u_j^{-1} v_j v_k^{-1} + v_i v_j^{-1} u_j^{-1} v_j v_k^{-1} = u_j^{-1} (-v_j + v_i) v_k^{-1} = 1$$

Let us now check that the gluing equations are respected. If the ideal triangulation of the knot complement contains an edge with only one preimage by p_I , then both R_I and R_H are the ring with one element and all conditions are trivially satisfied.

Before considering the general case, let us first remark the following. Let two triangles in \mathcal{T} be glued together along an oriented short edge e_j with initial point corresponding to long edges with shape parameters z_1 and z_2 . Let e_i, e_j, e_k be short edges clockwise ordered as in figure 4. Then, $f(z_1)f(z_2) = (v_i v_j^{-1})(v_j v_k^{-1}) = v_i v_k^{-1}$.

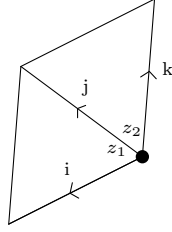


Figure 4: Glued triangles

Let us now consider separately few different cases. First, if an edge E in the ideal triangulation has no corresponding one in a distinguished tetrahedron of the H-triangulation, then the product of the $f(z_i)$ around E telescopically reduces to 1.

Second, E has a corresponding edge opposite to the distinguished edge in the distinguished tetrahedron. Then, after telescopication, keeping notations of lemma 2, the product of $f(z_i)$'s around E reduces to $v_{\tilde{m}}v_l^{-1}$ which is equal to 1 due to lemma 2.

Third case, E has a corresponding edge in the distinguished tetrahedron in the boundary of the distinguished face. Again, and after telescopication, the product of $f(z_i)$'s around E reduces to a product of parameters v_i which is equal to 1. In order to see this, it is easier to use the equivalent definition of $f : f(z) = u_p u_q^{-1}$ as in remark 3. Then, by using the notation of figure 2, the edge a_2 will contribute $u_{i_2} u_{j_2}^{-1}$ and the edge a_1 will contribute $u_{i_1}^{-1} u_{j_1}$ (both a_k appear twice in the distinguished tetrahedron). The product of those two contributions is 1 due to hexagonal face relations of figure 2.

Theorem 4 *If $H_1(M, \mathbb{Z}) = 0$, then f is a ring isomorphism.*

Remark 5 According to Poincaré duality and Universal coefficients theorem, if $H_1(M, \mathbb{Z}) = 0$, then $H_2(M, \mathbb{Z}) = 0$.

4 Proof of theorem 4

First of all, let us clear up the particular cases where the ideal triangulation of a knot complement contains an edge with only one preimage by p_I . Such an edge must be in a tetrahedron closed as a book around it. Then, the shape parameter of that edge must be equal to 1, which means that the deformation variety is empty. As a consequence, the ring R_I is the ring with one element. Besides, according to lemma 2, in that case, there is an invertible element in R_H equal to 0. So R_H is also the ring with one element and the isomorphism between R_I and R_H is trivial.

For the general case, let us first remark that the image of an element of R_I by f is a priori a product of an even number of elements in R_H . Then, in order to prove the surjectivity of f , the main difficulty is to write an equality in R_H with one generating element on the left hand side and an even number of

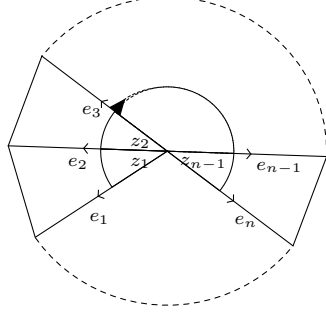


Figure 5: $\prod_{i=1}^{n-1} f(z_i) = v_1 v_n^{-1}$

generating elements on the right hand side. For that, we will use the triviality of $H_1(M, \mathbb{Z})$.

From now on, we will say that an oriented short edge $e \in \tilde{\mathcal{S}}$ is *parallel* (resp. *antiparallel*) to an oriented long edge E of $\tilde{\mathcal{L}}$ if there exists E' of $p_H^{-1}(p_H(E))$ such that e and E' are disjointly contained in the boundary of one and the same hexagonal face F , and if the orientations of \bar{e} (resp. e) and E' are induced from the orientation of F . We will also say that two oriented short edges e and e' are *parallel* (resp. *antiparallel*) if there exists a long edge E of $\tilde{\mathcal{L}}$ such that e and e' are both parallel or antiparallel (resp. one is parallel and another one is antiparallel) to E . We will also say that an edge $e \in \mathcal{S}$ is parallel (resp. antiparallel) to an edge $E \in \mathcal{L}$ if there exists a parallel (resp. antiparallel) preimage of e and E by p_H . For instance, in figure 2, m , c and l are parallel.

Lemma 6 *If e and e' are parallel short edges in \mathcal{S} , then there exists $m \in R_I$ such that $f(m) = u_e u_{e'}^{-1}$.*

Proof We will prove that there exists $m \in R_I$ such that $f(m) = v_{\check{e}} v_{\check{e}'}^{-1}$ (see subsection 2.2 for the definition of \check{e} and \check{e}'), which is equivalent to the lemma 6.

Let $E \in \mathcal{L}$ be parallel to e and e' . Then, among the triangles containing the edge E in $\overline{\mathcal{T}_H}$ there are some which also contain the edge \check{e} or \check{e}' . On the disk composed by the triangular faces containing a given end point of E , choose a path from \check{e} to \check{e}' . Let e_1, \dots, e_n be short edges met by this path and oriented so that they all have the same starting point (here $\check{e} = e_1$ and $\check{e}' = e_n$). Let $\tilde{E}_1, \dots, \tilde{E}_{n-1} \in p_H^{-1}(E)$ be long edges intersecting e_1, \dots, e_n in their starting point, and let z_1, \dots, z_{n-1} be the corresponding parameters in the ideal triangulation (see figure 5). Let $m = \prod_{i=1}^{n-1} z_i \in R_I$. According to the gluing equations in R_I , m does not depend on the choice of the path from \check{e} to \check{e}' . Besides, $\prod_{i=1}^{n-1} f(z_i) = \prod_{i=1}^{n-1} v_i v_{i+1}^{-1} = v_1 v_n^{-1} = v_{\check{e}} v_{\check{e}'}^{-1}$ (here, (u_i, v_i) is the couple of generators in R_H associated to the short edge e_i). \square

It will be usefull in the following to denote $m_{e \rightarrow e'}$ for such a word.

Corollary 7 *Let $e \in \mathcal{S}$ and $m \in R_I$ be such that $f(m) = u_e$. Then, for any short edge e' parallel to e , there exists $m' \in R_I$ such that $f(m') = u_{e'}$.*

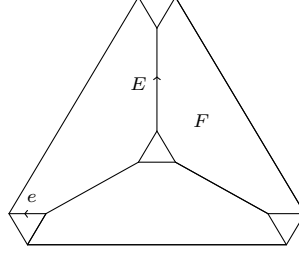


Figure 6: Definition of $r : e = r(F, E)$

Being a cell complex decomposition of M , \mathcal{T}_H induces a presentation of the first homology group with 1-cells as generators, and 2-cells as relations. Let us denote by K the distinguished edge corresponding to the knot in M . Let us also denote by E_i^H for $1 \leq i \leq n$ all the other edges in $\Delta^{1,H}/\Phi^H$ and $E_i^{j,H}$ the edges of $\Delta^{1,H}$ such that $p_H(E_i^{j,H}) = E_i^H$. Let us denote by $\Delta_l^H(M)$ the group of oriented l -chains (the free abelian group with basis the oriented open l -simplices), with boundary maps: $\partial_l^H : \Delta_l^H(M) \rightarrow \Delta_{l-1}^H(M)$. Let us denote by F^K the distinguished face of $\Delta^{2,H}/\Phi^H$, and F_i^H for $1 \leq i \leq m$ the other faces. Then $\partial_2^H(F^K) = \epsilon_K K + \epsilon_j E_j^H + \epsilon_{j'} E_{j'}^H$ ($\epsilon_i = \pm 1$). Besides, according to the particular H-triangulations considered here, F^K is the only face of $\Delta^{2,H}/\Phi^H$ whose boundary contains K . Thus, one can consider another presentation of the first homology group with generators the edges in $\Delta^{1,H}/\Phi^H$ without the distinguished one, and relations corresponding to non-distinguished faces. On what follows, we assume that K and F^K are removed from the corresponding sets of cells.

Let us assume from now on that $H_1(M, \mathbb{Z}) = 0$ and define some useful notations. Let A be an edge in $\Delta^{1,H}/\Phi^H$. Then, there exists a map $\eta_A : \Delta^{2,H}/\Phi^H \rightarrow \mathbb{Z}$ such that

$$A = \sum_{F \in \Delta^{2,H}/\Phi^H} \eta_A(F) \partial_2^H(F) \in \Delta_1^H(M).$$

It is a tautological equality in a free abelian group.

For any $F \in \tilde{\mathcal{H}}$ let

$$\tilde{\mathcal{E}}_F = \{E \in \tilde{\mathcal{L}} \mid E \text{ is on the boundary of } F\}$$

and let us define a set

$$X = \{(F, E) \mid F \in \tilde{\mathcal{H}}, E \in \tilde{\mathcal{E}}_F\}$$

and a map $r : X \rightarrow \tilde{\mathcal{S}}$ in the following way. E is on the boundary of two faces in $\tilde{\mathcal{H}}$, F and F' , so $r(F, E)$ is the short edge on the boundary of F' disjoint from E and parallel to it (see figure 6).

We fix a section $\sigma : \mathcal{H} \rightarrow \tilde{\mathcal{H}}$ with the image disjoint with the distinguished tetrahedron. Denote $\tilde{\mathcal{F}} = \sigma(\mathcal{H})$ and reinterpret η_A as a map from $\tilde{\mathcal{F}}$ to \mathbb{Z} .

We define

$$U_A = \prod_{F \in \tilde{\mathcal{F}}} \prod_{E \in \tilde{\mathcal{E}}_F} (u_{r(F,E)})^{\eta_A(F)}$$

Let us also define a map $s : \mathcal{L} \rightarrow \mathcal{S}$ which associates arbitrarily, to each long edge, a parallel short edge, and

$$U_A^s = \prod_{F \in \tilde{\mathcal{F}}} \prod_{E \in \tilde{\mathcal{E}}_F} (u_{s \circ p_H(E)})^{\eta_A(F)}$$

and remark that $U_A^s = u_{s(A)}$; it is a tautological equality in the free abelian group generated by the symbols u_i associated to short edges. Then, for each $A \in \Delta^{\mathbf{1}, \mathbf{H}} / \Phi^{\mathbf{H}}$, one can write the equality

$$u_{s(A)} = \frac{U_A^s}{U_A} = \prod_{F \in \tilde{\mathcal{F}}} \prod_{E \in \tilde{\mathcal{E}}_F} \left(\frac{u_{s \circ p_H(E)}}{u_{r(F,E)}} \right)^{\eta_A(F)}.$$

By definitions of r and s and thanks to lemma 6, there exist words $m_{F,E} \in R_I$ such that $\frac{u_{s \circ p_H(E)}}{u_{r(F,E)}} = f(m_{F,E})$ and thus there exists a word $m_{s(A)}$ in R_I such that $f(m_{s(A)}) = u_{s(A)}$.

According to corollary 7, for any short edge e_i parallel to $s(A)$ and with the generators of R_H (u_i, v_i), there exists a word $m_i \in R_H$ such that $f(m_i) = u_i$. Besides, for any short edge e_j , there exists a short edge e_i such that $v_j = u_i$ or $v_j = u_i^{-1}$. Thus f is surjective, and this almost allows one to define a ring homomorphism $g : R_H \rightarrow R_I$: keeping the previous notation, $g(u_i) = g(v_j) = m_i$.

One still has 4 things to check:

1. The images of the u_i do not depend on the choice of η_A :

Let $\eta_A^{(1)}$ and $\eta_A^{(2)}$ be two maps such that

$$A = \sum_F \eta_A^{(1)}(F)F = \sum_F \eta_A^{(2)}(F)F \in \Delta_1^H(M).$$

As $H_2(M, \mathbb{Z}) = 0$, $\phi = \sum_F \eta_A^{(1)}(F)F - \sum_F \eta_A^{(2)}(F)F = \partial_3^H(\tau)$, where τ is a linear combination of tetrahedra in \mathcal{T}_H . If τ contains the distinguished tetrahedron, then there is an ambiguity in the word $m_{s(A)}$ given by a power of the product of all shape parameters associated to the edge opposite to the distinguished edge in the distinguished tetrahedron, which is 1 thanks to the gluing equations. Let us now assume that τ contains only tetrahedra different from the distinguished one. Let us denote by $m_{s(A)}^{(1)}$ and $m_{s(A)}^{(2)}$

the corresponding words for $\eta_A(1)$ and $\eta_A(2)$. Then $\frac{m_{s(A)}^{(1)}}{m_{s(A)}^{(2)}}$ is the product of all shape parameters of all tetrahedra contained in τ , so it is equal to 1.

2. The definition of U_A does not depend on the choice of $\tilde{\mathcal{F}}$ i.e. the section $\sigma : \mathcal{H} \rightarrow \mathcal{H}$:

Let $F \in \mathcal{H}$ and $\{F^1, F^2\} = p_H^{-1}(F)$. If one of the F^i 's is on the boundary of the distinguished tetrahedron, then there is no problem thanks to the

definition of σ . Else, let $E_k^i, 1 \leq k \leq 3, 1 \leq i \leq 2$ be the long edges in the boundary of F^i , with the corresponding shape parameters z_k^i . Let $e_k^i = r(F^i, E_k^i)$. It is then easy to see that for each k , $f(z_k^1 z_k^2) = u_k^1 (u_k^2)^{-1}$, then the ambiguity of $m_{s(A)}$ is given by a power of $\prod_{k=1}^3 z_k^1 z_k^2 = 1 \in R_I$.

3. The definition of m_i does not depend on the choice of s :

Let $A \in \mathcal{L}$, $e_1, e_2 \in \mathcal{S}$ with parameters (u_1, v_1) , (u_2, v_2) , and $s_1, s_2 : \mathcal{L} \rightarrow \mathcal{S}$ such that $s_1(A) = e_1$, $s_2(A) = e_2$ and for any $E \in \mathcal{L}, E \neq A$, $s_1(E) = s_2(E)$. Then, from the definition of g , one can write $g_{s_1}(u_1) = m_{s_1(A)}$, $g_{s_2}(u_2) = m_{s_2(A)}$ and $g_{s_2}(u_1) = m_{e_1 \rightarrow e_2} m_{s_2(A)}$.

The sum of the powers of u_i in the definition of $U_A^{s_i}$ is 1, so $U_A^{s_1} = \frac{u_1}{u_2} U_A^{s_2}$ and

$$\begin{aligned} g_{s_1}(u_1) &= m_{s_1(A)} \\ &= m_{e_1 \rightarrow e_2} m_{s_2(A)} \\ &= g_{s_2}(u_1). \end{aligned}$$

The same reasoning works for $g_{s_1}(u_2) = g_{s_2}(u_2)$.

For $E \neq A$, the sum of the powers of u_i in the definition of $U_E^{s_i}$ is 0, so changing s has no influence on the definition of $g(u)$ for the short edges parallel with E .

4. Let us now prove that g respects the relations in R_H .

The hexagonal face relations are preserved by definition of the image of v_i .

Until now, orientation of long edges, hence the sign, was implied in the notation, and short edges were considered parallel with long edges. Reversing orientation of a short edge naturally inverses the image of its generator u by g .

Let us check now, that reversing orientation preserves the relation for the parameter v .

Let e_1, e_2, e_3 be three short edges on the boundary of a face $F \in \tilde{\mathcal{H}}$, respectively parallel to long edges E_1, E_2, E_3 (also on the boundary of F) and oriented such that the starting point of e_1 and e_3 bound E_2 , and the end point of e_1 and e_2 bound E_3 . Then, by definition of g , $g(v_2) = g(u_1)$. Let us reverse orientation of e_2 and let us call it \bar{e}_2 , then $g(v_{\bar{2}}) = g(u_3)$, and one has to check that $g(u_3) = -g(u_2)^{-1} g(u_1)$. Let us do this from the definition of $g(u_3)$ and let us assume without loss of generality, that $e_3 = s(E_3)$. Let us write $\partial_2^H(F) = E_2 + E_3 - E_1 \in \Delta_1^H(M)$, so

$$\begin{aligned} E_3 &= \partial_2^H(F) + E_1 - E_2 \\ &= \partial_2^H \left(F + \sum_{\phi} \eta_{E_1}(\phi) \phi - \sum_{\phi} \eta_{E_2}(\phi) \phi \right). \end{aligned}$$

Let us assume, as a first case, that $p_H(E_1), p_H(E_2), p_H(E_3)$ are three different edges and that $e_1 = s(E_1)$ and $e_2 = s(E_2)$. Let us denote

$u'_i = r(F, E_i)$. Then, one has the following tautological equality:

$$u_3 = \frac{u_3 u'_1 u_2}{u'_3 u_1 u'_2} \frac{U_1^s}{U_1} \frac{U_2}{U_2^s}.$$

$\frac{u_3 u'_1 u_2}{u'_3 u_1 u'_2}$ has as preimage in R_I a product of three different shape parameters of a tetrahedron (which is equal to -1), $\frac{U_1^s}{U_1}$ has as preimage in R_I a word m_1 from lemma 6 such that $f(m_1) = u_1$ and $\frac{U_2^s}{U_2}$ has as preimage in R_I a word m_2 from lemma 6 such that $f(m_2) = u_2$. So, by construction of g , $g(u_3) = -g(u_1)g(u_2)^{-1}$. Let us now assume that another edge of the boundary of F has projection equal to E_3 in $\Delta^{1, \mathbf{H}}/\Phi^{\mathbf{H}}$, for example E_1 , with $s(E_1) = s(E_3) = e_3$. Then, one has the following tautological equality: $u_3 = \frac{u_3 u'_1 u_2}{u'_3 u_1 u'_2} \frac{U_1^s}{U_1} \frac{U_2}{U_2^s}$, but now, $\frac{U_1^s}{U_1} = u_3$ and $\frac{u_3 u'_1 u_2}{u'_3 u_1 u'_2} = \frac{u_3 u'_1 u_2}{u'_3 u_1 u'_2} \frac{u_1}{u_3}$ which has preimage in R_I the product of three different shape parameters of a tetrahedron and the word of corollary 7 going from e_3 to e_1 .

Let us end with the triangular face relations. Let e_i, e_j, e_k be short edges bounding a triangular face in $\overline{\mathcal{T}_H}$, with generators $(u_i, v_i), (u_j, v_j), (u_k, v_k)$. Let F be the opposite hexagonal face, with long edges E_i, E_j, E_k on its boundary. Let us assume they are oriented so that $\partial_2^H(F) = E_i + E_j - E_k$, and one has to prove that $g(u_i)g(u_j)g(u_k)^{-1} = 1$: the reasoning is similar to the previous one. One also has to prove that

$$-g(v_k) + g(u_i)g(v_j) + g(v_i) = 0,$$

or equivalently,

$$g(u_i)g(v_j)g(v_k)^{-1} + g(v_i)g(v_k)^{-1} = 1,$$

so, by definition of g ,

$$g(u_i)g(u_j)g(u_k)^{-1} + g(u_i)g(u_k)^{-1} = 1.$$

Let us call e_l , the short edge in the same tetrahedron parallel with e_i , then it is equivalent to prove that

$$g(u_i)g(u_l)^{-1}g(u_l)g(u_j)g(u_k)^{-1} + g(u_i)g(u_k)^{-1} = 1.$$

The left part of the equality is equal, by definition of g and thanks to the preceding relation, to $-z_k z_j + z_j = 1$.

It is now a straightforward computation to check that $f \circ g = id_{R_I}$ and $g \circ f = id_{R_H}$.

Remark 8 We actually proved that for any edge E of $\Delta_1^H(M)$ of which the projection onto $H_1(M)$ is trivial, the parameters u_i associated to short edges parallel to E are in the image of f .

5 Examples

5.1 Figure-eight knot

Let K be the figure-eight knot embedded in \mathbb{S}^3 . The cell complex of figure 7 is a truncated particular H-triangulation corresponding to the well known ideal

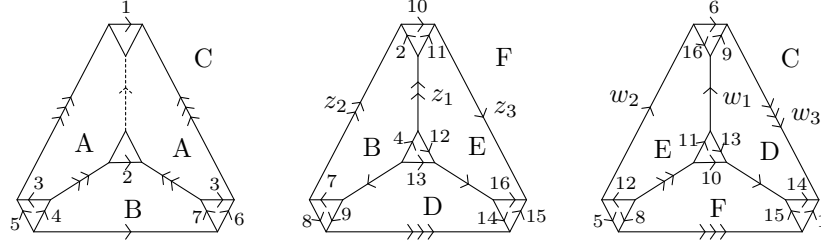


Figure 7: Figure eight knot in \mathbb{S}^3

triangulation of $\mathbb{S}^3 \setminus K$, see [3]. The distinguished edge is dotted. The edges labelled with double and triple arrows are identified when collapsing that edge onto the vertex before truncation. The shape parameters z_i and w_i are actually associated to the corresponding edges of the ideal triangulation.

Thurston's gluing equations are $z_1^2 z_2 w_1 w_3^2 = 1$ and $z_2 z_3^2 w_1 w_2^2 = 1$, or, equivalently, $z_1(1 - z_1)w_3(1 - w_3) = 1$.

Keeping the previous notations, one has, for example, $f(z_1) = v_2 v_{11}^{-1} = u_7 u_{16}^{-1}$. According to theorem 4, f is an isomorphism. Let us see, for example, what the preimage of u_1 is. e_1 is parallel to the edge labelled with a simple arrow. That edge is tautologically equal to $\partial_2^H(B) \in \Delta_1^H(\mathbb{S}^3)$. Let us consider the only triangular face opposite to the hexagonal face corresponding to B which is not in the distinguished tetrahedron. Let us assume that the image by s of the long edge labelled with a double arrow is e_{15} , and e_1 for the long one labelled with a simple arrow. Then, one has the following equality : $u_1 = \frac{u_1 u_{15} u_{15}^{-1}}{u_{14} u_{15} u_{16}}$. Besides, following the proof of lemma 6, one has $f(w_2 z_2 w_2 z_3 w_1) = \frac{u_1}{u_{14}}$ and, by definition of f and thanks to the hexagonal face relations in R_H , $f(w_3) = \frac{u_{16}}{u_{15}}$. Then, using the gluing equations in R_I , we find that $f(z_2 z_3 w_1 w_2^2 w_3) = u_1$.

5.2 A knot in $\mathbb{S}^2 \times \mathbb{S}^1$

Figure 8 represents a truncated particular H-triangulation of a knot in $\mathbb{S}^2 \times \mathbb{S}^1$. The knot is represented by the dotted edge. Here, $M = \mathbb{S}^2 \times \mathbb{S}^1$, so $H_1(M, \mathbb{Z}) \simeq H_2(M, \mathbb{Z}) \simeq \mathbb{Z}$.

The shape parameters z and w are associated to the corresponding edges in \mathcal{T}_I . The Thurston's gluing equation is $zw = 1$. After computation, the presentation of R_H reduces to $\mathbb{Z}\langle u_1, u_{\bar{1}}, u_2 | u_1 u_{\bar{1}} = 1, u_2^3 = 1 \rangle$

f is neither surjective, nor injective. Indeed, $u_1 \notin \text{Im}(f)$, and $f(w) = u_2$, so $f(w^3) = 1$.

5.3 Knot in $L(3, 1)$

The figure 9 represents a truncated particular H-triangulation of a knot in $L(3, 1)$. The knot is represented by the dotted edge. The shape parameters z, t, u, w are actually associated to edges of the corresponding ideal triangulation. Let us remind that $H_1(L(3, 1), \mathbb{Z}) = \mathbb{Z}/3\mathbb{Z}$ and $H_2(L(3, 1), \mathbb{Z}) = 0$.

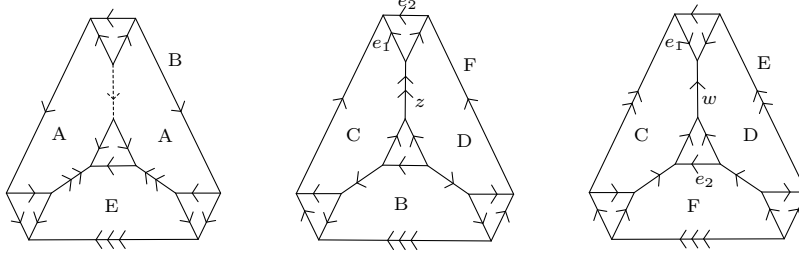


Figure 8: Knot in $\mathbb{S}^2 \times \mathbb{S}^1$. All edges are oriented, but for simplicity, only the necessary names have been given.

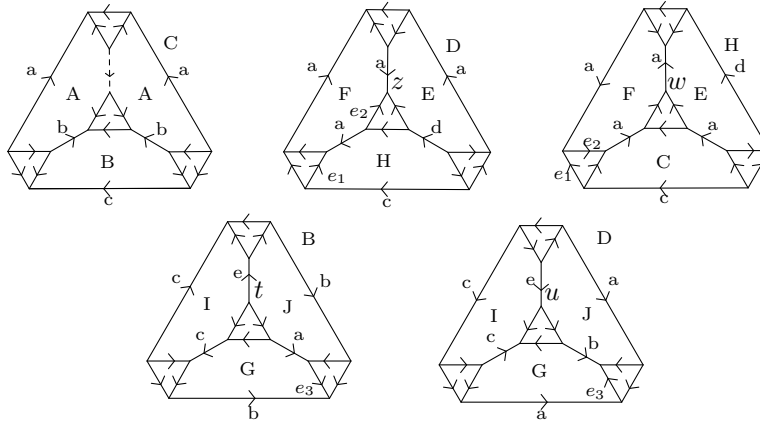


Figure 9: Knot in $L(3,1)$. All edges are oriented, but for simplicity, only the necessary names have been given.

The gluing equations are $t = u^{-1}$ and $z = w^{-1}$ and the presentation of the ring R_H reduces to $\mathbb{Z}\langle u_1, u_{\bar{1}}, u_2, u_3, u_{\bar{3}} \mid u_1 u_{\bar{1}} = 1, u_3 u_{\bar{3}} = 1, u_2^3 = -1 \rangle$. f is not surjective, because $\text{Im}(f) = \mathbb{Z}\langle u_1 u_2^2, u_{\bar{1}} u_2, u_3, u_{\bar{3}} \mid u_1 u_{\bar{1}} = 1, u_3 u_{\bar{3}} = 1, u_2^3 = -1 \rangle$.

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